11970. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Let

$$\zeta_5(z) = 1 + 2^{-z} + 3^{-z} + 4^{-z} + 5^{-z},$$

where z is a complex number. Prove that $\zeta_5(z) \neq 0$ when the real part of z is greater than or equal to 0.9.

SOLUTIONS

Successors of Squares Without Large Prime Divisors

11831 [2015, 390]. *Proposed by Raitis Ozols, University of Latvia, Riga, Latvia.* Prove that for $\varepsilon > 0$ there exists an integer n such that the greatest prime divisor of $n^2 + 1$ is less than εn.

Solution by John P. Robertson, National Council on Compensation Insurance, Boca Raton, Florida. We show more generally that, for every integer k and every $\varepsilon > 0$, there exists an integer n such that the greatest prime divisor of $n^2 + k$ is less than εn.

When $k = 0$, it suffices to let n be a sufficiently large power of 2.

When $k \neq 0$, let $n = 4k^2m^3 + 3km$ for some positive integer m. We compute

$$n^2 + k = k(km^2 + 1)(4km^2 + 1)^2.$$

The largest prime p dividing $n^2 + k$ can be no larger than $|4km^2 + 1|$. In addition, $|4km^2 + 1|/n < \varepsilon$ when m is sufficiently large, so $p < \varepsilon n$.

Editorial comment. Some solutions involved the factorization of $x^{420} - 1$ into cyclotomic polynomials, and some used Pell's equation.

Also solved by D. Beckwith, B. Bekker (Russia) & Y. J. Ionin, R. Chapman (U. K.), V. De Angelis, J. Hosle, P. W. Lindstrom, O. P. Lossers (Netherlands), W. McDermott, M. Omarjee (France), L. Robitaille, C. P. Rupert, J. Schlosberg, N. C. Singer, R. Stong, R. Tauraso (Italy), E. Weinstein, M. Wildon (U. K.), and the proposer.

An Inequality When Two Triples Agree in Order

11834 [2015, 390]. *Proposed by Arkady Alt, San Jose, CA.* For nonnegative real numbers u, v, w, let $\Delta(u, v, w) = 2(uv + vw + wu) - (u^2 + v^2 + w^2)$. Say that two lists (a, b, c) and (x, y, z) *agree in order* if $(a - b)(x - y) \geq 0$, $(b - c)(y - z) \geq 0$, and $(c - a)(z - x) \geq 0$. Prove that if (x, y, z) and (a, b, c) agree in order, then $\Delta(a, b, c)\Delta(x, y, z) \geq 3\Delta(ax, by, cz)$.

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA. The quantity Δ is invariant under permutation of its arguments. The relation "agree in order" and the inequality to be proved do not change when (a, b, c) and (x, y, z) undergo the same permutation. Therefore we may assume $a \geq b \geq c \geq 0$ and $x \geq y \geq z \geq 0$. One then sees that

$$\begin{aligned}\Delta(a, b, c)\Delta(x, y, z) - 3\Delta(ax, by, cz) &= (a - b)^2\big(4(x - y)^2 + 3(y - z)^2\big)\\ &+ 3(b - c)^2(x - y)^2 + 2(a - b)^2\big((x - y)(3y - z) + z(y - z)\big)\\ &+ 2(a - b)\big((x - y)^2(3b - c) + (x - y)(3by - cz) + cz(y - z)\big)\\ &+ 2c(b - c)\big((x - y)^2 + z(x - z) + z(y - z)\big)\end{aligned}$$

is nonnegative, since each term is nonnegative.

Also solved by R. Chapman (U. K.), H. Y. Far, J. F. Loverde, L. Matejíčka (Slovakia), J. C. Smith, R. Stong, S. Wagon, and the proposer.